CLASS – 11

WORKSHEET- SYSTEMS OF PERICLES AND ROTATIONAL MOTION

A. CENTRE OF MASS

(1 Mark Questions)

1. Define centre of mass.

Sol. A point of which the entire mass of the body on system of bodies is supposed to be concentrated is known as centre of mass.

2. The velocity of centre of mass of the system remains constant, if the total external force acting on the system is

(a) minimum (b) maximum (c) unity (d) zero

Sol. (d)

3. Show that the centre of mass of an isolated system moves with a uniform velocity along a straight line path.

Sol. Let vector M be the total mass concentrated at centre of mass whose position vector is vector r.

$$\vec{F} = \frac{Md^2\vec{r}}{dt^2}$$

$$\vec{F} = \frac{Md}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{Md}{dt}\left(\overrightarrow{Vcm}\right)$$

For an isolated system $\vec{F} = 0$

$$\Rightarrow \frac{Md}{dt}\left(\overrightarrow{Vcm}\right) = 0$$

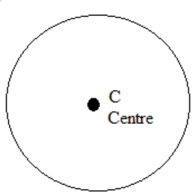
$$or \ \frac{d}{dt}\left(\overrightarrow{Vcm}\right) = 0 \ as \ M \neq 0$$

$$\Rightarrow \left(\overrightarrow{Vcm}\right) = \text{constant}$$

4. For which of the following does the centre of mass lie outside the body?

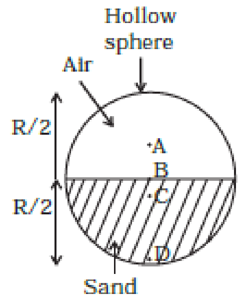
(a) A pencil (b) A shotput (c) A dice (d) A bangle

Sol. (d)



A bangle is in the form of a ring as shown in figure. The centre of mass of the ring lies at its centre, which is outside the ring or bangle.

5. Which of the following points is the likely position of the centre of mass of the system shown in Fig.?



(a) A (b) B (c) C (d) D

Sol. (c) C

Centre of mass of a system lies upward the part of the system, having bigger mass. In the above diagram, lower part is heavier, hence CM of the system lies below the horizontal diameter at C point.

(2 Marks Questions)

6. Two point mass of 1kg and 2kg lie at (1, 2) and (2, - 3) respectively. Calculate the coordinates of the centre of mass of the system.

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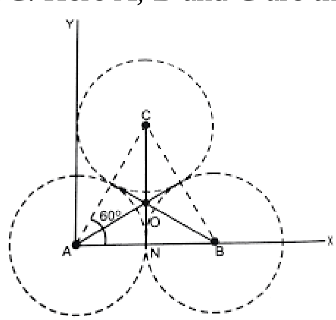
7. Three identical spheres each of radius ‘r’ and mass ‘m’ are placed on a vertical plane such that each spheres touching each other and stay in equilibrium. Find the position of centre of mass.

Sol. The centre of mass of each sphere is at its geometrical centre. SO the centre of mass of the system is in fact the centre of mass of three equal point masses located at the vertices of an equilateral triangle ABC. Here A, B and C are the centres of three spheres.



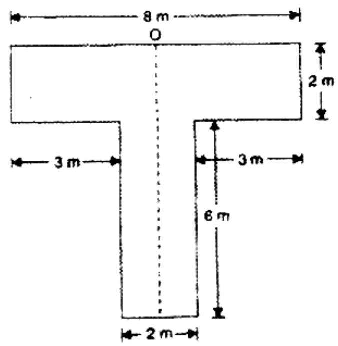
Position of centre of mass

$x_{CM} = \frac{m\times r+m\times 0+m\times 2r}{3m} = r$

$y_{CM} = \frac{3\times r\sqrt{3}+m\times 0+m\times 0}{3m} = \frac{r}{\sqrt{3}}$

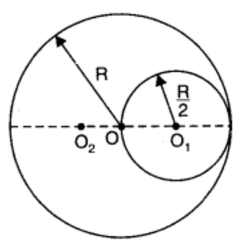
$\therefore (x_{CM}, y_{CM}, z_{CM}) = \left(r, \frac{r}{\sqrt{3}}\right)$

8. Find the position of the centre of mass of the T-shaped plate from O in figure.



Sol. Let mass per unit area of the plate be σ.
Mass of horizontal portion = 8 × 2σ = 16σ, Mass of vertical portion = 6 × 2σ = 12σ
The centres O_1 and O_2 of these portions lie at distances 1m and 2+3 = 5m from the point O. The CM of the T shaped plate will lie at distance y from the point IO which is given by $y = \frac{16\sigma\times 1+12\sigma\times 5}{16\sigma+12\sigma} = \frac{76\sigma}{28\sigma} = 2.71$m.

9. From a uniform disk of radius R, a circular hole of radius R/2 is cut out. The centre of the hole is at R/2 from the centre of the original disc. Locate the centre of gravity of the resulting flat body.



Sol. The situation is shown in figure. Let O be the CM of the original circular position. O_1 that of the circular hole cut out and O_2 that of the remaining portion. Let m be the mass per unit area of the disc.

Mass of original disc, $M = \pi R^2 m$

Mass of the circular hole cut, $m_1 = \pi\left(\frac{R}{2}\right)^2 m = \frac{\pi}{4}R^2 m$

Mass of the remaining portion, $m_2 = \pi R^2 m - \frac{\pi}{4}R^2 m = \frac{3}{4}\pi R^2 m$

Masses m_1 and m_2 may be assumed to be concentrated at O_1 and O_2 respectively and O is their CM.

Therefore, Moment of m_1 about O = Moment of m_2 about O

Or $m_1 \times O_1O = m_2 \times O_2O$

Or $\frac{\pi}{4}R^2m \times \frac{R}{2} = \frac{3}{4}\pi R^2m \times O_2O$

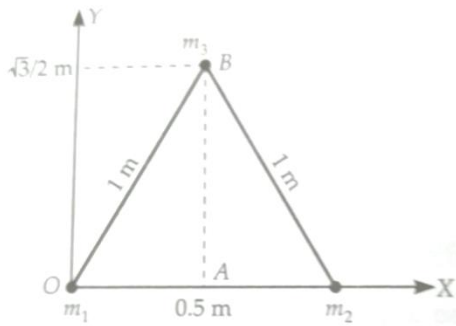
Or $O_2O = R/6$

Thus CM of the resulting potion lies at R/6 from the centre of the original disc in a direction opposite to the centre of the cut out portion.

(3 Marks Questions)

10. Three masses 3, 4 and 5kg are located at the corners of an equilateral triangle of side 1m. Locate the centre of mass of the system.

Sol. Suppose the equilateral triangle lies in the XY plane with mas 3kg at the origin. Let (x, y) be the coordinates of CM.



Clearly, $AB = \sqrt{OB^2 - OA^2} = \sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}m$

Now $x_1 = 0$, $x_2 = 1m$, $x_3 = OA = 0.5m$, $m_1 = 3kg$, $m_2 = 4kg$, $m_3 = 5kg$.

$\therefore x = \frac{m_1x_1+m_2x_2+m_3x_3}{m_1+m_2+m_3} = \frac{3\times0+4\times1+5\times0.5}{3+4+5} = \frac{6.5}{12} = 0.54m$

Again $y_1 = 0$, $y_2 = 0$, $y_3 = AB = \frac{\sqrt{3}}{2}$

$\therefore y = \frac{m_1y_1+m_2y_2+m_3y_3}{m_1+m_2+m_3} = \frac{3\times0+4\times1+5\times(\frac{\sqrt{3}}{2})}{3+4+5} = \frac{5\times\sqrt{3}}{2\times12} = 0.36m$

Thus the coordinates of CM are (0.54m, 0.36m).

11. Two bodies of masses 10kg and 2kg are moving with velocities $2\hat{i} - 7\hat{j} + 3\hat{k}$ and $-10\hat{i} + 35\hat{j} - 3\hat{k}\ ms^{-1}$ respectively. Find the velocity of centre of mass of the system.

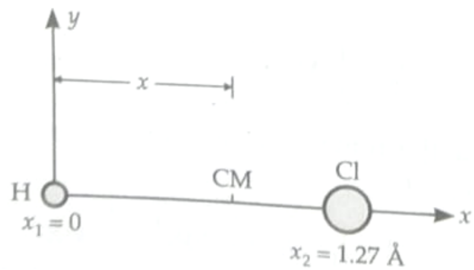
Sol. Velocity of Centre of mass = $\frac{m_1v_1+m_2v_2}{m_1+m_2} = \frac{10(2i-7j+3k)+2(-10i+35j-3k)}{10+12} = 2\hat{k}\ ms^{-1}$.

12. Give the location of the centre of mass of a (i) sphere, (ii) cylinder, (iii) ring, and (iv) cube, each of uniform mass density. Does the centre of mass of a body necessarily lie inside the body?

Sol. (i) Geometrical centre. (ii) Centre of its axis of symmetry. (iii) Centre of the ring. (iv) Point of intersection of the diagonals.
No, it is not necessary that centre of mass of a body lies inside the body. For example, in a ring, hollow sphere in a tumbler, etc., the centre of mass lies inside their hollow portion.

13. In the HCl molecule, the separation between the nuclei of the two atoms is about 1.27 Å (1 Å = 10^{-10}m). Find the approximate location of the CM of the molecule, given that a chlorine atom is about 35.5 times as massive as a hydrogen atom and nearly all the mass of an atom is concentrated in its nucleus.

Sol. A shown in figure, suppose the H nucleus is located at the origin. Then, $x_1 = 0$, $x_2 = 1.27$ Å, $m_1 = 1$, $m_2 = 35.5$



The position of the CM of HCl molecule is $x = \frac{m_1x_1+m_2x_2}{m_1+m_2} = \frac{1\times0+35.5\times1.27}{1+35.5}$ = 1.235 Å.
Thus the CM of HCl is located on the line joining H and Cl nuclei at a distance of 1.235 Å from the H nucleus.

14. A child sits stationary at one end of a long trolley moving uniformly with a speed V on a smooth horizontal floor. If the child gets up and runs about on the trolley in any manner, what is the speed of the CM of the (trolley + child) system?

Sol. The forces involved in the given problem are the internal forces of the system. No external force acts on the system when the child runs. So there will be no change in the speed of the centre of mass of the (trolley + child) system.

B. INTRODUCTION OF ROTATORY MOTION

(1 Mark Questions)

1. A flywheel rotating at 420 rpm shows down at a constant rate of 2 rad/s. The time required to stop the flywheel is
(a) 22s (b) 11s (c) 44s (d) 12s

Ans. (a)

2. What is a rigid body?

Sol. A rigid body is a solid body in which deformation is zero or so small it can be neglected.

3. When a disc rotates with uniform angular velocity, which of the following is not true?
(a) The sense of rotation remains same.

(b) The orientation of the axis of rotation remains same.
(c) The speed of rotation is non-zero and remains same.
(d) The angular acceleration is non-zero and remains same.

Sol. (d)
Angular acceleration, $\alpha = d\omega/dt$ where ω is angular velocity of the disc and is uniform or constant. $\alpha = d\omega/dt = 0$. Hence angular acceleration is zero.

(2 Marks Questions)

4. What is the value of linear velocity, if $\vec{r} = 3\hat{i} + 4\hat{j} + 6\hat{k}$ and $\vec{\omega} = -5\hat{i} + 3\hat{j} + 5\hat{k}$?

Sol. Here $\vec{\omega} = -5\hat{i} + 3\hat{j} + 5\hat{k}$, $\vec{r} = 3\hat{i} + 4\hat{j} + 6\hat{k}$

$$\vec{v} = \vec{w} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5 & 3 & 5 \\ 3 & 4 & 6 \end{vmatrix}$$

$= \hat{i}\,(18 - 20) - \hat{j}\,(-30 - 15) + \hat{k}\,(-20 - 9)$

$= -2\hat{i} + 45\hat{j} - 29\,\hat{k}$

(3 Marks Questions)

5. A constant power is supplied to a rotating disc. How is angular velocity (ω) of disc varies with number of rotations (n) made by the disc?

Sol. We have $P = \tau.\omega$

Or $P = I\left(\omega \frac{d\omega}{d\theta}\right)\omega$ or $\omega^2 d\omega = \frac{P}{I} d\theta$

On integrating,

$\int_0^\omega \omega^2 d\omega = \int_0^\theta \frac{P}{I} d\theta \Rightarrow \left[\frac{\omega^3}{\omega}\right]_0^\omega = \frac{P}{I}[\theta]_0^\theta$

$\Rightarrow \left[\frac{\omega^3}{\omega} - 0\right] = \frac{P}{I}[\theta - 0]; \omega^3 = \frac{3P}{I}\theta$

We find that $\omega \propto (\theta)^{1/3}$ or $\omega \propto (n)^{1/3}$.

(5 Marks Questions)

6. Define rotational motion of a body. Derive the following equations of rotational motion under constant angular acceleration.

(a) $\omega = \omega_0 + \alpha t$ (b) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ (c) $\omega^2 = \omega_0^2 + 2\alpha\theta$

Sol. A body is said to possess rotational motion if all the particles move along circles in parallel planes. The centres of these circles lie in a fixed line perpendicular to the parallel planes and this line is called the axis of rotation.

(a) Derivation of first equation of motion: Consider a rigid body rotating about a fixed axis with constant angular acceleration α. By definition,

$\alpha = \frac{d\omega}{dt}$

Or $d\omega = \alpha\, dt$ …(1)

At t= 0, let $\omega = \omega_0$, At t = t, let $\omega = \omega$

Integrating equation (1) within the above limits of time and angular velocity, we get

$\int_{\omega_0}^{\omega} d\omega = \int_0^1 \alpha dt = \alpha \int_0^1 dt$

Or $[\omega]_{\omega_0}^{\omega} = \alpha[t]_0^t$

Or ω □ $\omega_0 = \alpha(t - 0)$

Or $\omega = \omega_0 + \alpha t$ …(2)

(b) Deviation of second equation of motion: Let ω be the angular velocity of a rigid body at any instant t. By definition, $\omega = \frac{d\theta}{dt}$

Or $d\theta = \omega\, dt$ …(3)

At t= 0, let θ = 0; At t = t, let θ = θ

Integrating equation (3) within the above limits of time and angular displacement, we get

$\int_0^{\theta} d\theta = \int_0^t \omega dt = \int_0^t (\omega_0 + \alpha t)dt$ [using (2)] = $\omega_0 \int_0^t dt + \alpha \int_0^t t\, dt$

Or $[\theta]_0^{\theta} = \omega_0[t]_0^t = \alpha\left[\frac{t^2}{2}\right]_0^t$

Or $\theta - 0 = \omega_0(t - 0) + \frac{\alpha}{2}(t^2 - 0)$

Or $\theta = \omega_0 t + ½\, \alpha t^2$ …(4)

(c) Deviation of third equation of motion: The angular acceleration α may be expressed as $\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}.\frac{d\theta}{dt} = \frac{d\omega}{d\theta}.\omega$

Or $\omega\, d\omega = \alpha\, d\theta$ …(5)

At t = 0, θ = 0 and $\omega = \omega_0$ (initial angular velocity)

At t = t, θ = θ and ω = ω (final angular velocity)

Integrating equation (5) within the above limits of θ and ω we get

$\int_{\omega_0}^{\omega} \omega d\omega = \int_0^{\theta} \alpha d\theta = \alpha \int_0^{\theta} d\theta$

Or $\left[\frac{\omega^2}{2}\right]_{\omega_0}^{\omega} = \alpha[\theta]_0^{\theta}$

Or $\frac{\omega^2}{2} - \frac{\omega_0^2}{2} = \alpha(\theta - 0)$

Or $\omega^2 - \omega_0^2 = 2\alpha\theta$

C. MOMENT OF INERTIA

(1 Mark Questions)

1. The moment of inertia of a body depends upon
   (a) mass of the body (b) axis of rotation of the body
   (c) shape and size of the body (d) all of these

Sol. (d)

2. Define 1 kg m^2.

Sol. 1 kg m^2 is the moment of inertia about a fixed axis when mass of the particle constituting the body is 1kg an distance of the particle from the axis of rotation of the body is 1m.

3. Two masses each of mass M are attached to the end of a rigid massless rod of length L. The moment of inertia of the system about an axis passing through centre of mass and perpendicular to its length is

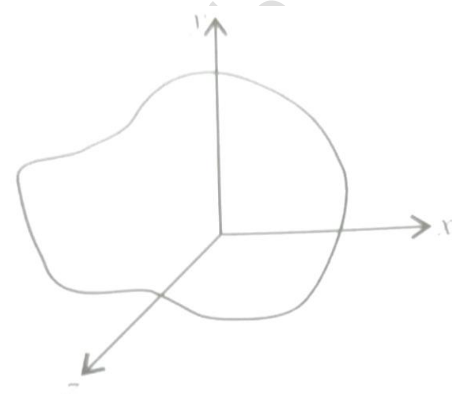
(a) $ML^2/4$ (b) $ML^2/2$ (c) ML^2 (d) $2ML^2$

Sol. (b)

4. State the theorem of perpendicular axes.

Sol. The moment of inertia of a planar body about an axis (z) perpendicular to its plane is equal to the sum of its plane is equal to the sum of its moment of inertia about two perpendicular axes (x) and (y) concurrent with the perpendicular axis and lying in the plane of the body.

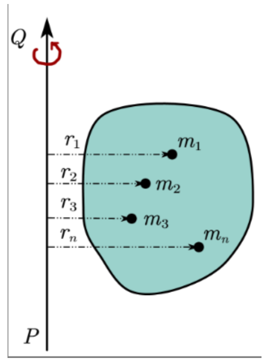
$I_z = I_x + I_x$



(2 Marks Questions)

5. Define radius of gyration of a body rotating about an axis. Derive an expression for it.

Sol. The radius of gyration of a body about a given axis is the distance of the point from the axis of rotation where if the whole of the mass of the body were concentrated, it would have the same moment of inertia as it has with the actual distribution of mass. It is denoted by k.



Suppose a rigid body consists of n particles of mass m each, situated at distance r_1, r_2, r_3., …r_n from the axis of rotation PQ is $I = mr_1^2 + mr_2^2 + mr_3^2 + \ldots + mr_n^2$

$= m[r_1^2 + r_2^2 + r_3^2 + \ldots r_n^2]$

$= m \times n \frac{[r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2]}{n}$ (where M = m × n = total mass of the body)

If k is the radius of gyration about the axis AB, then $I = Mk^2$

Therefore $Mk^2 = M\left(\frac{r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2}{n}\right)$

$k = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2}{n}}$ = Root mean square distance

6. If two circular discs A and B are of same mass but of radii r and 2r respectively, then what is the moment of inertia of A in terms that of B?

Sol. We know that MI of disc, $I = MR^2/2$

Therefore $I \propto R^2$

$\therefore \frac{I_A}{I_B} = \frac{M_A}{M_B}\left(\frac{R_A}{R_B}\right)^2 = 1\left(\frac{1}{2}\right)^2 = \frac{1}{4}$

Therefore $I_A = I_B$

7. State parallel axis theorem of moment of inertia. What is the moment of inertia of a (i) ring (ii) disc about diameter?

Sol. The moment of inertia of a body about an axis parallel to the body passing through its centre is equal to the sum of the moment of inertia of the body about the axis passing through the centre and the product of the mass of the body times the square of the distance of between the two axes.

(i) The moment of inertia of a disc about any of its diameter is $MR^2/4$.

(ii) the mass moment of inertia of a thin circular ring is $I = MR^2$.

8. Three mass points m_1, m_2 and m_3 are located at the vertices of an equilateral triangle of length a. What is the moment of inertia of the system about an axis along the altitude of the triangle passing through m_1?

Sol. Moment of inertia is the product of mass and square of separation between particle and axis of rotation .

e.g , M.I=mr²
here, we see, separation of mass m1 and altitude NN′is 0 .
alteration between mass m_2 and NN′ is (a/2) also for m_3 separation is (a/2)
Moment of inertia about altitude passing through $m_1=I_1+I_2+I_3$
where I_1, I_2, and I_3 are M.I of m_1, m_2 and m_3 respectively .
M.I= $m_1.(0) + m_2(a/2)^2 + m_3(a/2)^2$
$= \frac{a^2}{4(m_2+m_3)}$

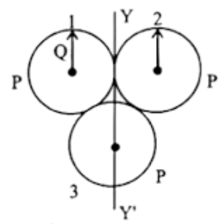
9. State the factors on which the moment of inertia of a body depends.

Sol. **The moment of inertia depends on the following:**

- Mass of the body.
- Size and shape of the body.
- Distribution of mass about the axis of rotation.
- Position and orientation of the axis of rotation with respect to the body.

(3 Marks Questions)

10. Three identical rings, each of mass M and radius R are arranged as shown in figure. What is the moment of inertia of the arrangement about YY'?

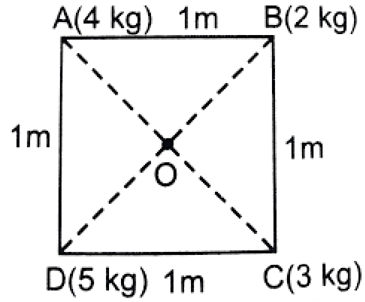


Sol. Moment of inertia of ring I about YY' = ½ MR^2 + MR^2 = 3/2 MR^2
Moment of inertia of ring II about YY' = ½ MR^2 + MR^2 = 3/2 MR^2
Moment of inertia of ring III about YY' = ½ MR^2
Moment of inertia of the system about YY' = 3/2 MR^2 + 3/2 MR^2 + ½ MR^2 = 7/2 MR^2

11. Calculate the MI of a uniform circular disc of mass 500gm and radius 10cm about (i) Diameter (ii) Axis tangent to the disc and parallel to diameter (iii) Axis passing through centre and perpendicular to its plane.

Sol. Mass of the disc. M = 500g = 0.5kg, Radius of the disc, R = 10cm = 0.1m
(i) Moment of inertia of a uniform circular disc about its diameter
I_d = ¼ MR^1 = ¼ × 0.5 × $(0.1)^2$ = 1.25 × 10^{-3}kg m^2
(ii) Moment of inertia of the disc about axis tangent to the disc and parallel to diameter.
I = I_d × MR^2 = ¼ MR^2 + MR^2 = 5/4 MR^2 or I = 6.45 × 10^{-3} kg m^2
(iii) Moment of inertia of the disc about axis passing through centre and perpendicular to its plane.
I = ½ MR^2 = 2.5 × 10^{-3} kg m^2

12. Four particles of masses 4kg, 2kg, 3kg and 5kg are respectively located at the four corners A, B, C and D of a square of side 1m as shown in figure. Calculate the moment of inertia of the system about (i) an axis passing through the point of intersection of the diagonals and perpendicular to the plane of the square. (ii) the side AB and (iii) the diagonal BD.



Sol. Here AB = BC = CD = DA = 1m, OA = OB = OC = OD = $\frac{1}{2}\sqrt{1^2+1^2} = \frac{1}{\sqrt{2}}$m

(i) MI of the system about an axis through O and perpendicular to the plane of the square,

I = $4(OA)^2 + 2(OB)^2 + 3(OC)^2 + 5(OD)^2 = (4+2+3+5) \times \left(\frac{1}{\sqrt{2}}\right)^2$ = 14 × ½ = 7 kg m^2

(ii) MI of the system about the side AB, I = $3(BC)^2 + 5(AD)^2$ = 3 × 1 + 5 × 1 = 8 kg m^2

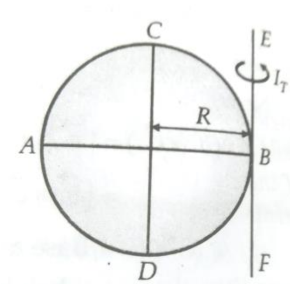
(iii) MI of the system about the diagonal BD, I = $4(OA)^2 + 3(OC)^2$ = 4 × ½ + 3 × ½ = 3.5 kg m^2.

13. Three particles, each of 10g are located at the corners of an equilateral triangle of side 5cm. Determine the moment of inertia of this system about an axis passing through one corner of the triangle and perpendicular to the plane of the triangle

Sol. Moment of inertia due to mass passing through the axis is zero.
$2mr^2 = 2 \times 10^{-2} \times 25 \times 10^{-4} = 5 \times 10^{-5}$ kgm^2.

14. (a) Find the moment of inertia of a sphere about a tangent to the sphere, given the moment of inertia of the sphere about any of its diameters to be 2 MR^2/5, where M is the mass of the sphere and R is the radius of the sphere.
(b) Given the moment of inertia of a disc of mass M and radius R about any of its diameters to 1 be 1/4 MR^2, find the moment of inertia about an axis normal to the disc passing through a point on its edge.

Sol. (a) Here $I_D = 2/5\ MR^2$



The tangent EF is parallel to the diameter CD. By the theorem of parallel axis, $I_{EF} = I_{CD} + MR^2$

$I_T = 2/5\ MR^2 + MR^2 = 7/5MR^2$

(b) By the theorem of parallel axes, MI about an axis passing through an edge of t\point and normal to the disc = MI about central normal axis + MR^2

$I' = ½\ MR^2 + MR^2 = 3/2\ MR^2$.

D. ROTATIONAL DYNAMICS

(1 Mark Questions)

1. The SI unit of angular momentum is
(a) kg ms^{-1} (b) Nm (c) kg m^2s^{-1} (d) Nm^2

Ans. (c)
The SI unit of angular momentum is kg m^2s^{-1}

2. When a torque acting upon a system is zero. Which of the following will be constant?
(a) Force (b) Linear impulse (c) Linear momentum (d) angular momentum

Ans. (d)

3. A rigid body is said to be in partial equilibrium, when it is in
(a) transitional equilibrium only (b) rotational equilibrium only
(c) either (a) or (b) (d) neither (a) nor (b)

Ans. (c)
A rigid body is said to be in partial equilibrium when it is in translational equilibrium but not in rotational equilibrium or when it is in rotational equilibrium but not in translational equilibrium.

4. A disc is rotating with angular velocity $\vec{\omega}$ about its axis. A force $\vec{F}$ acts at a point whose position vector with respect to the axis of rotation is $\vec{r}$. The power associated with the torque die the force is given by
(a) $(\vec{r}\times\vec{F}).\vec{\omega}$ (b) $(\vec{r}\times\vec{F})\times\vec{\omega}$ (c) $\vec{r}.(\vec{F}\times\vec{\omega})$ (d) $\vec{r}\times(\vec{F}\times\vec{\omega})$

Ans. (a)

5. Which of the following principles a circuit acrobat employs in his performance?
(a) Conservation of energy (b) Conservation of linear momentum
(c) Conservation of mass (d) Conservation of angular momentum

Ans. (d)

6. State right hand rule to find the direction of angular momentum.

Ans. Curl the fingers of the right hand in the direction of rotation, then the thumb points in the direction of angular momentum.

7. Why do we prefer to use wrench of longer arm?

Sol. The torque applied on the nut by the wrench is equal to the force multiplied by the perpendicular distance from the axis of rotation. Hence to increase torque a wrench of longer arm is preferred.

8. Is it difficult to open the door by pushing it or pulling it at the hinge. Why?

Sol. When the force is applied at the hinges, the line of action of the force passes through the axis of rotation, i.e., r = 0 so $\tau = rF \sin\theta = 0$. So, we cannot open the door by pushing or pulling it at the hinges.

9. Why a force is applied at right angles to the heavy door at the outer edge while closing or opening it?

Sol. Torque, $\tau = rF \sin\theta$. For a force applied at right angle to the outer edge of the door both $\sin\theta$ (= $\sin 90° = 1$) and r are maximum. Hence the torque produced is maximum.

10. A faulty balance with unequal arms has its beam horizontal. Are the weights of the two pans equal?

Sol. They are of unequal mass. Their masses are in the inverse ratio of the arms of the balance.

11. Which physical quantities are expressed by the following: (i) the rate of change of angular momentum, and (ii) moment of linear momentum?

Sol. (i) Torque (ii) Angular momentum.

12. If the earth were to shrink suddenly, what would happen to the length of the day?

Sol. When the earth shrinks, the moment of inertia ($I = 2/5\ MR^2$) decreases about its own axis due to the decrease in radius R. To conserve angular momentum, ($L = I\omega = I.\frac{2\pi}{T}$), the time period T decreases. That is, the length of the day increases.

13. Two identical particles move towards each other with velocity 2v and v respectively. What is the velocity of the centre of mass?

Sol. Here $m_1 = m_2 = m$; $v = 2v$ and $v_2 = -v$.

$\therefore v_{CM} = \frac{m_1v_1+m_2v_2}{m+m} = \frac{m\times2v+m(-v)}{m+m} = \frac{v}{2}$.

14. What is torque? Give its SI unit.

Sol. The turning effect of a force about the axis of rotation is called of moment of force or torque due to the force.

Torque = Force × its perpendicular distance from the axis of rotation
Or τ = Force × moment arm = F.
Its SI unit is Newton-meter or kgm^2sec^{-2}.

15. Which physical quantity is represented by the product of moment of inertia and angular velocity?

Sol. Moment of inertia of a body is the rotational inertia of the body. It is the rotational analogue of mass in linear motion.

16. Define the term moment of momentum.

Sol. It is equal to the product of linear momentum and the perpendicular distance of its line of action from the axis of rotation.

17. A Merry-go-round, made of a ring-like platform of radius R and mass M, is revolving with angular speed ω. A person of mass M is standing on it. At one instant, the person jumps off the round, radially away from the centre of the round (as seen from the round). The speed of the round afterwards is

(a) 2ω (b) ω (c) $\omega/2$ (d) 0

Sol. (a)

As there is external torque acting on the system, angular momentum should be conserved.
Hence, $I\omega$ = constant …(i)
Where I is moment of inertia of the system and ω is angular velocity of the system.
Form eq. (i), $I_1\omega_1 = I_2\omega_2$ where w_1 and w_2 are angular velocities before and after jumping.
(since $I = mr^2$) so, given that, $m_1 = 2M$, $m_2 = M$, $\omega_1 = \omega$, $\omega_2 = ?$, $r_1 = r_2 = R$, $m_1r_1^2\omega_1 = m_2r_2^2\omega_2$
$2MR^2w = MR^2\omega_2$ as ,ass reduced to half, hence moment of inertia also reduced to half.

(2 Marks Questions)

18. If $\vec{r} = 2\hat{i} + 3\hat{j}$ and $\vec{F} = 4\hat{i} - 3\hat{j}$, then find the magnitude of torque.

Sol. $\vec{r} = 2\hat{i} + 3\hat{j}$ and $\vec{F} = 4\hat{i} - 3\hat{j}$

$$\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 0 \\ 4 & -3 & 0 \end{vmatrix}$$

$= \hat{i}\,[0] - \hat{j}\,[0] + \hat{k}\,[-6 - 12] = -18\hat{k}$.

Hence magnitude of $\vec{\tau} = |\vec{r} \times \vec{F}| = |-18\hat{k}| = 18$.

19. The position of a particle is given by $\vec{r} = \hat{i} + 2\hat{j} - \hat{k}$ and its linear momentum is given by $\vec{p} = 3\hat{i} + 4\hat{j} - 2\hat{k}$. In which axis its angular momentum, about the origin is perpendicular?

Sol. $\vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 3 & 4 & -2 \end{vmatrix}$

$= \hat{i}\,(-4 + 4) - \hat{j}\,(-2 + 3) + \hat{k}\,(4 - 6) = 0\hat{i} - 1\hat{j} - 2\hat{k}$

$\vec{L}$ has components along y axis and – z axis but it has no component along the x axis. The angle of momentum in y z plane i.e. perpendicular to x axis.

20. How a ballet dancer does take the advantage of the principle of conservation of angular momentum?

Sol. During the course of their performance a ballet dancer take advantage of the principle of conservation of angular momentum i.e., $I\omega$ = constant, by stretching out arms and legs or vice versa. In doing so, their moment of inertia increases or decreases. Hence angular velocity w of their spin motion decreases or increases accordingly.

21. A solid sphere rolls down an inclined plane. Find the ratio of its rotational kinetic energy, the total kinetic energy.

Sol. Translational kinetic energy, $E_T = \frac{1}{2} mv^2$

Rotational kinetic energy, $E_R = \frac{1}{2} I\omega^2$

Total energy, $E = E_T + E_R = \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2$

$= \frac{1}{2} mv^2 + \frac{1}{2} \times 2/5\, mr^2 \times v^2/r^2$

$= \frac{1}{2} mv^2 + 1/5\, mv^2 = 7/10\, mv^2$

$E_R/E_T = (1/5\, mv^2)/(7/10 mv^2) = 2/7 \Rightarrow E_R/E_T = 2/7$

22. Find the torque of force $7\hat{i} - 3\hat{j} - 5\hat{k}$ about the origin which acts on a particle whose position vector is $\hat{i} + \hat{j} - \hat{k}$.

Sol. Here $\vec{F} = 7\hat{i} - 3\hat{j} - 5\hat{k}$ and $\vec{r} = \hat{i} + \hat{j} - \hat{k}$

$\therefore\ \vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 7 & -3 & -5 \end{vmatrix}$

Or $\vec{\tau} = -8\hat{i} - 2\hat{j} - 10\hat{k}$.

23. To maintain a rotor at a uniform angular speed of 200 rad s^{-1}, an engine needs to transmit a torque of 180 Nm. What is the power required by the engine?

Sol. Here $\omega = 200$ rad s^{-1}, $\tau = 180$Nm

Therefore, Power, $P = \tau\omega = 180 \times 200 = 36{,}000$W = 36 kW.

24. A metre stick is balanced on a knife edge at its centre. When two coins, each of mass 5g are put one on top of the other at the 12.0 cm mark, the stick is found to be balanced at 45.0 cm. What is the mass of the metre stick?

Sol. As the metre stick balances at 50.0 cm mark, its CG must lie at this mark.

The 45.01cm mark is the CG of the metre stick + 2 coins system.
Let m be the mass of the meter stick.
Distance between 50.0 cm mark and new CG = 50.0 – 45.0 = 5.0 cm
Distance between 12.0 cm mark and new CG = 45.0 – 12.0 = 33.0 cm
From the principle of moments (for equilibrium), mg × 5.0 = (2 × 5) × g × 33.0
Or $m = \frac{2\times5\times33.0}{5.0} = 66.0g$

(3 Marks Questions)

25. Define torque. Derive an expression for it in Cartesian coordinates.

Sol. We know that $\vec{\tau} = \vec{r} \times \vec{F}$

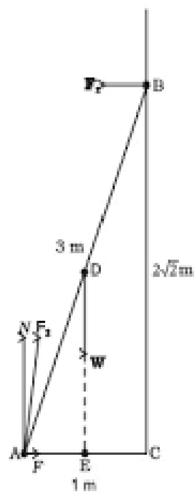
If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$, we have

$$\vec{\tau} = \tau_x\hat{i} + \tau_y\hat{j} + \tau_z\hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$= \hat{i}\,(yF_z - zF_y) - \hat{j}\,(xF_z - zF_x) + \hat{k}\,(xF_y - yF_x)$

Comparing the coefficient of $\hat{i}$, $\hat{j}$ and $\hat{k}$ we have $\tau_z = xF_y - yF_x$.

26. A 3m long ladder weighing 20kg leans on a frictionless wall. Its feet rest on the floor 1m from the wall as shown in figure. Find the reaction forces of the wall and the floor.



Sol. The ladder AB is 3m long, its foot A is at distance AC = 1m from the wall.

Using Pythagoras theorem, BC = $\sqrt{AB^2 - AC^2} = \sqrt{3^2 - 1^2} = 2\sqrt{2}$m

The forces acting on the ladder are:

1. Its weight W acting on the centre of gravity D.

2. Reaction force F_1 of the wall. It acts perpendicular to the wall because the wall is frictionless.

3. Reaction force F_2 of the floor. This floor can be resolved into two components: the normal reaction N and the force of friction f. The friction prevents the ladder sliding away from the wall and hence acts towards the wall.
For translational equilibrium, balancing the forces in the vertical direction, N = W
Balancing the forces in the horizontal direction, $f = F_1$
For rotational equilibrium, we consider the moments about the point A.
Clockwise moment = Anticlockwise moment, $F_1 \times BC = W \times AE$
Or $F_1 \times 2\sqrt{2} = W \times ½$
But W = 20g = 20 × 9.8 = 196.0N
∴ N = W = 196.0N
$F_1 = \frac{W}{4\sqrt{2}} = \frac{196.0}{4\sqrt{2}} = 34.6N$
$f = F_1 = 34.6N$ and $F_2 = \sqrt{f^2 + N^2} = \sqrt{34.6^2 + 196^2} = 199.0N$
If the force F_2 makes angle α with the horizontal, then $\tan\alpha = \frac{N}{f} = 4\sqrt{2} = 5.6568$
Therefore α = 80°.

27. A flywheel of mass 25kg has a radius of 0.2m. What force would be applied tangentially to the rim of the flywheel so that it acquires an angular acceleration of 2 rad s^{-1}?

Sol. Here M = 25kg, R = 0.2m, α = 2 rad s^{-2}
MI of the flywheel about its axis, $I = ½ MR^2 = ½ \times 25 \times (0.2)^2 = 0.5$ kg m^2
As torque, τ = F, R = Iα
So, Force, $F = \frac{I\alpha}{R} = \frac{0.5\times2}{0.2} = 5N$

28. A grindstone has moment of inertia of 6 kg m^2. A constant torque is applied and the grindstone is found to have a speed o f150 rpm, 10 seconds after starting from rest. Calculate the torque.

Sol. Here I = 6 kg m^2, t = 10s, $\omega_0 = 0$, v = 150rpm = 150/60 rps = 5/2 rps, ω = 2πv = 2π × 5/2 = 5π rad s^{-1},
$\alpha = \frac{\omega-\omega_0}{t} = \frac{5\pi-0}{10} = \frac{\pi}{2}$rad s^{-2}
Torque, $\tau = I\alpha = 6 \times \frac{\pi}{2} = 3\pi$ Nm.

29. A ring of diameter 0.4m and mass 10kg is rotating about its axis at the rate of 2100 rpm. Find (i)moment of inertia (ii) angular momentum and (iii) rotational KE of the ring.

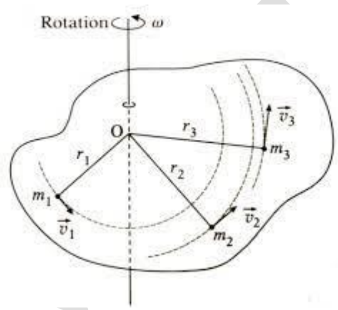
Sol. Here R = 0.4/2 = 0.2m, M = 10kg, v = 2100rpm = 2100/60 =rps = 35s^{-1}.
∴ ω = 2πv = 2 × 22/7 × 35 = 220 rad s^{-1}
(i) MI of the ring about its axis, $I = MR^2 = 10 \times (0.2)^2 = 0.4$ kg m^2.
(ii) Angular momentum, L = Iω = 0.4 × 220 = 88 kg m^2s^{-1}
(iii) Rotational KE = $½ I\omega^2 = ½ \times 0.4 \times (220)^2 = 9680J$

30. Explain if the ice on the polar caps of the earth melts, how will it affect the duration of the day?

Sol. If the ice on the polar caps of the earth melts, the water so formed will spread on the surface of the earth. This increases the moment of inertia (I) of the earth about its own axis (due to change in the distribution of mass of the particles of water going away from the axis of rotation). To conserve angular momentum (= Iω), ω (angular velocity of the earth about its own axis) will decrease. As $T = 2\pi/\omega$, hence due to decrease in the value of ω, T i.e., the duration of the day will increase.

31. Establish the relation between torque and angular acceleration. Hence define moment of inertia.

Sol. Relation between torque and moment of inertia: When a torque acts on a body capable of rotation about its axis, it produces an angular acceleration in the body. If the angular velocity of each particle is ω, then the angular acceleration, $\alpha = d\omega/dt$ will be same for all the particles of the body. The linear acceleration will depend on heir distances r_1, r_2=, …r_n from the axis of rotation.



As shown in figure, consider a particle P of mass m_1 at a distance $r_1$from the axis of rotation. Let its linear velocity be v_1.

Linear acceleration of the first particle, $a_1 = r_1\alpha$

Force acting on the first particle, $F_1 = m_1r_1\alpha$

Moment of inertia F_1 about the axis of rotation is $\tau_1 = F_1r_1 = m_1r_1^2\,\alpha$

Total torque acting on the rigid body is

$\tau = \tau_1 + \tau_2 + \tau_3 + \ldots\tau_n$

$= m_1r_1^2\alpha + m_2r_2^2\,\alpha + m_3r_3^2\,\alpha + \ldots. + m_nr_n^2\,\alpha$

$= (m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + m_nr_n^2)\alpha$

$= (\Sigma\, mr^2)\alpha$

But $\Sigma\, mr^2 = I$, moment of inertia of the body about the given axis, $\tau = I\alpha$

Torque = Moment of inertia × Angular acceleration

When $\alpha = 1$, $\tau = 1$

Thus moment of inertia of a rigid body about an axis of rotation is numerically equal to the external torque required to produce unit angular acceleration in the body about that axis.

32. State and prove the principle of conservation of angular momentum.

Sol. Law of conservation of angular momentum: Suppose the external torque acting on a rigid body due to external forces is zero. Then $\tau = dL/dt = 0$.

Here L = constant

So, when the total external torque acting on a rigid body is zero, the total angular momentum of the body is conserved. This is the law of conservation of angular momentum.

Clearly, when $\tau = 0$. $L = I\omega$ = constant

Or $I_1\omega_1 = I_2\omega_2$

This means that when no external torque is acting, the angular velocity ω of the body can be increased or decreased by decreasing or increasing the moment of inertia of the body.

33. Find the components along the x, y, z-axes of the angular momentum $\vec{\mathbf{l}}$ of a particle, whose position vector is $\vec{\mathbf{r}}$ with components x, y, z and momentum is $\vec{\mathbf{p}}$ with components p_x, p_y and p_z. Show that if the particle moves only in the x-y plane the angular momentum has only a z- component.

Sol. We can write $\vec{\mathbf{l}} = l_x\hat{i} + i_y\hat{j} + i_z\hat{k}$

$\vec{\mathbf{r}} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{\mathbf{p}} = p_x\hat{i} + p_y\hat{j} + p_z\hat{k}$

But $\vec{\mathbf{l}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$

$= (x\hat{i} + y\hat{j} + z\hat{k}) \times (p_x\hat{i} + p_y\hat{j} + p_z\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

Or $l_x\hat{i} + l_y\hat{j} + l_z\hat{k}$

$= \hat{i}\,(yp_z - zp_y) + \hat{j}\,(zp_x - xp_z) + \hat{k}\,(xp_y - yp_x)$

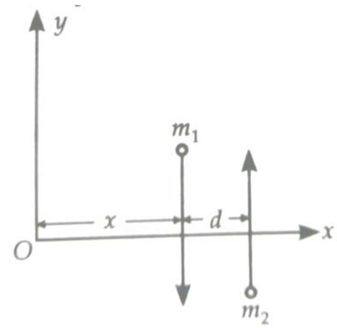
Comparing the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ on the two sides, we et the components of $\vec{\mathbf{l}}$ as follows: $l_x = yp_z - zp_y$; $l_y = zp_x - xp_z$ and $l_z = xp_y - yp_x$

If the particle is constrained to move only in the xy plane, then z = 0 and $p_z = 0$. Hence $\vec{\mathbf{l}} = \hat{k}\,(xp_y - yp_x)$

As only the unit vector $\hat{k}$ corresponding to z direction survives, the angular momentum $\vec{\mathbf{l}}$ has only a z component.

34. Two particles, each of mass m and speed v, travel in opposite directions along parallel lines separated by a distance d. Show that the vector angular momentum of the two particle system the same whatever be the point about which the angular momentum is taken.

Sol. As shown in figure, suppose the two particles move parallel to the y axis.



Total angular momentum of the two particles system about O is

$\vec{\mathbf{l}} = \vec{\mathbf{l}}_1 + \vec{\mathbf{l}}_2 = \vec{\mathbf{r}}_1 \times \vec{\mathbf{p}}_1 + \vec{\mathbf{r}}_2 \times \vec{\mathbf{p}}_2$

$= x\hat{i} \times (-mv)\,\hat{j} + (x+d)\,\hat{i} \times (mv)\,\hat{j}$

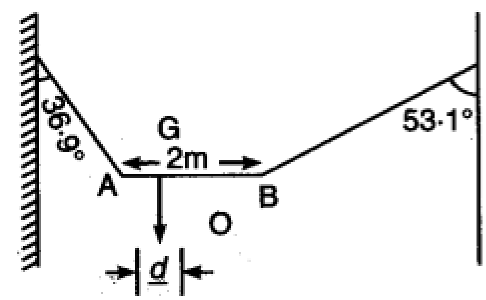
$= (-mvx)\,\hat{i} \times \hat{j} + (mvx + mvd)\,\hat{i} \times \hat{j}$

$= (-mvx + mvx + mvd)\,\hat{k}$

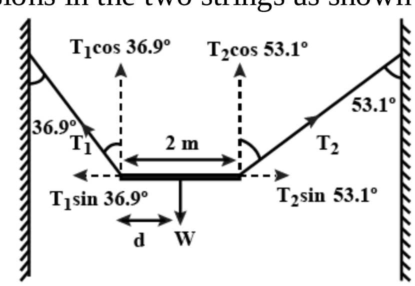
$= mvd\ k$ [since $\hat{i} \times \hat{j} = \hat{k}$]

Clearly $\vec{\mathbf{l}}$ does not depend on x and hence on the origin O. Thus the angular momentum of the two particle system is same whatever be point about which the angular momentum is taken.

35. A non-uniform bar of weight W is suspended at rest by two strings of negligible weight as shown in Fig. The angles made by the strings with the vertical are 36.9° and 53.2° respectively. The bar is 2 m long. Calculate the distance d of the centre of gravity of the bar from its left end.



Sol. Let T_1 and T_2 be the tensions in the two strings as shown in figure.



For translational equilibrium, on balancing the vertical components of forces, we get

$T_1\cos\theta + T_2\cos\phi = W$ …(1)

On balancing the horizontal components, we get,

$T_1\sin\theta = T_2\sin\phi$ …(2)

For rotational equilibrium, we balance he torques about the CG of the bar.

Clockwise torque = Anticlockwise torque

$T_1\cos\theta \times d = T_2 \cos\phi \times (2 - d)$ …(3)

Dividing (3) by (2) we et

$d \cot \theta = (2- d) \cot \phi$

or $d \cot 36.9° = (2 - d) \cot 53.1°$

or $d \cot 36.9° = (2 - d) \tan 36.9°$

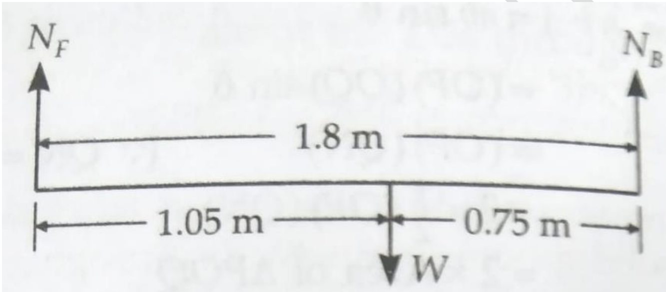
or $d \times 4/3 = (2 - d) \times ¾$

or $16d = 18 - 9d$

or $d = 18/25 = 0.72m = 72cm$

36. A car weighs 1800 kg. The distance between its front and back axles is 1.8 m. Its centre of gravity is 1.05 m behind the front axle. Determine the force exerted by the level ground on each front wheel and each back wheel.

Sol. Let N_B and N_F be the total reaction forces exerted by the level ground on front and back wheels respectively. The situation is shown in figure.



For translational equilibrium of the car,

$N_F + N_B = W = 1800 \times 9.8N$

Or $N_F + N_B = 17640N$

For rotational equilibrium of the car,

$1.05N_F = 0.75 N_B$

Or $1.05N_F = 0.75(17640 - N_F)$

Or $1.8N_F = 13230$

Or $N_F = 13230/1.8 = 7350N$

And $N_B = 17640 - 7350 = 10290$ N

Force on front wheel = 7350/2 = 3675N

Force on back wheel = 10290/2 = 5145N

37. Torques of equal magnitude are applied to a hollow cylinder and a solid sphere, both having the same mass and radius. The cylinder is free to rotate about its standard axis of symmetry, and the sphere is free to rotate about an axis passing through its centre. Which of the two will acquire a greater angular speed after a given time?

Sol. Let M and R be mass and radius of the hollow cylinder and the solid sphere. Then

MI of hollow cylinder about its axis of symmetry, $I_1 = MR^2$

MI of solid sphere about an axis through its centre, $I_2 = \frac{2}{5}MR^2$

Let α_1 and α_2 be angular accelerations produced in the rotational motion of the cylinder and the sphere on applying torque τ in each case. Then

$\alpha_1 = \frac{\tau}{I_1} = \frac{\tau}{MR^2}$

And $\alpha_2 = \frac{\tau}{\frac{2}{5}MR^2} = 2.5\frac{\tau}{MR^2} = 2.5\alpha_1$

As $\alpha_2 > \alpha_1$ and $\omega = \omega_0 + \alpha t$, so the solid sphere will acquire a greater angular speed after a given time.

38. A solid cylinder of mass 20 kg rotates about its axis with angular speed 100 rad s^{-1}. The radius of the cylinder is 0.25 m. What is the kinetic energy associated with the rotation of the cylinder? What is the magnitude of angular momentum of the cylinder about its axis?

Sol. Here M = 20kg, ω = 100 rad s^{-1}, R = 0.25m

MI of the cylinder about its own axis, $I = ½ MR^2 = ½\ 20 \times (0.25)^2 = 0.625$ kgm^2

Rotational KE = $½ I\omega^2 = ½ \times 0.625 \times (100)^2 = 3125J$.

Angular momentum, $L = I\omega = 0.625 \times 100 = 62.5$ kg m^2 s^{-1}.

39. A rope of negligible mass is wound round a hollow cylinder of mass 3 kg and radius 40 cm. What is the angular acceleration of the cylinder if the rope is pulled with a force of 30 N? What is the linear acceleration of the rope? Assume that there is no slipping.

Sol. Here M = 3kg, R = 40cm = 0.40m, F = 30N,

Torque, $\tau = F \times R = 30 \times 0.40 = 12Nm$

MI of the hollow cylinder about its own axis, $I = MR^2 = 3 \times (0.40)^2 = 0.48$ kg m^2

Angular acceleration, $\alpha = \frac{\tau}{I} = \frac{12}{0.48} = 25$ rad s^{-2}.

Linear acceleration, $a = R\alpha = 0.40 \times 25 = 10$ ms^{-2}.

40. The oxygen molecule has a mass of 5.30 x 10^{-26} kg and a moment of inertia of 1.94 x 10^{-45} kg m^2 about an axis through its centre perpendicular to the lines joining the two atoms. Suppose the mean speed of such a molecule in a gas is 500 m/s and that its kinetic energy of rotation is two thirds of its kinetic energy of translation. Find the average angular velocity of the molecule.

Sol. Rotational KE = $\frac{2}{3}$ Translational KE

Or $\frac{1}{2}I\omega^2 = \frac{2}{3}.\frac{1}{2}\ mv^2$

Or $\omega = v \times \sqrt{\frac{2m}{3I}} = 500\times\sqrt{\frac{2\times5.30\times10^{-26}}{3\times1.94\times10^{-46}}}$

$= 500 \times \sqrt{1.82 \times 10^{20}} = 500 \times 1.35 \times 10^{10}$ rad s^{-1}

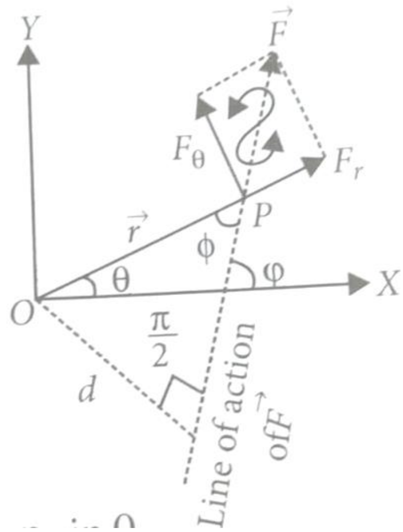
$= 6.75 \times 16^2$ rad s^{-1}.

(5 Marks Questions)

41. (a) Derive an expression for torque on polar coordinates.

(b) A torque of 20 Nm is applied on a wheel initially are rest. Calculate the angular momentum at the wheel after 3s.

Sol. (a)



Let a single particle whose position with reference to the origin O of the two dimensional coordinates system is given by position vector $\vec{r}$. Let O be the angle between OP and x axis and the location of the particle be P(x, y). Suppose the force $\vec{F}$ applied on the particle at P(x, y) then angle made by the applied force with positive direction of x axis is ϕ

Therefore $F_x = F \cos\phi$ and $F_y = F \sin\phi$

Now torque, $\tau = yF_x + xF_y = - yF \cos\phi + xF\sin\phi$

But $x = r \cos\theta$ and $y = r \sin\theta$

Therefore $\tau = r \sin\theta (\cos\phi) + (r \cos\theta)(f \sin\phi)$

Or $\tau = rF \sin\phi \cos\theta - rF \cos\phi \sin\theta$

Or $\tau = rF(\sin\phi\cos\theta - \cos\phi\sin\theta) = rF\sin(\phi$ □ $\theta)$ or $F\sin\phi$ [since $\sin(A - B) = \sin A\cos B - \cos A \sin B$ and $\theta + \phi = \phi - \theta$]

As d -= r sinϕ [d = perpendicular distance of the line of action of the force from the origin O]

Therefore $\tau = Fd$

The force F at P can be resolved into two rectangular components: radial and tangential. The radial component has no turning effect, so this is irrelevant. However, the rectangular component F_0 is perpendicular to radial component F_r, is responsible for turning effect.

$F_0 = F \sin(\phi$ □ $\theta) \therefore \tau = rF_0$ or $\tau = F_0 r$

So, torque is the product of transverse component of force and the distance from the axis of rotation.

(b) As $\vec{\tau} = \frac{d\vec{L}}{dt} = \therefore d\vec{L} = \vec{\tau}dt$

In magnitude, $dL = \tau dt$

As wheel is initially at rest, then $\int_0^L d\vec{L} = \int_0^3 \tau dt$ or $|\vec{L}| = 20 \times 3 = 6$ kg m^2 s^{-1}.

42. A flywheel of mass 25 kg has a radius of 0.2m. It is making 240rpm. What is the torque necessary to bring it to rest tin 20s? If the torque is due to a force applied tangentially on the rim of the flywheel, what is the magnitude of the force?

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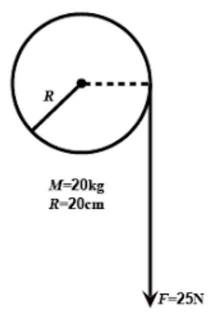
43. A cord of negligible mass is wound round the rim of flywheel of mass 20kg and radius 20cm. A steady pull of 25N is applied on the cord as shown in figure. The flywheel is mounted on a horizontal axis with frictionless bearings.



(a) Compute the angular acceleration of the wheel.
(b) Find the work done by the pull, when 2m of the cord is unwind.
(c) Find also the kinetic energy of the wheel at this point. Assume that the wheel starts from rest.
(d) Compare answers to parts (b) and (c).

Sol. (a) Torque, $\tau = FR = 25N \times 0.20m = 5.0$ Nm

Moment of inertia of the wheel about its axis, $I = \frac{MR^2}{2} = \frac{20\times(0.20)^2}{2} = 0.4$ kgm^2

As $\tau = I\alpha$

Therefore $\alpha = \frac{\tau}{I} = \frac{5.0Nm}{0.4kg\ m^2} = 12.5$ rad s^{-2}.

(b) Work done by the pull unwinding 2m on the cord = 25N × 2m = 50J.

(c) Angular displacement of the wheel, $\theta = \frac{\text{Length of unwound string}}{\text{Radius of the wheel}} = \frac{2\text{m}}{0.20\text{m}}$ = 10rad

As the wheel starts from rest, $\omega_0 = 0$

Final angular velocity ω is given by $\omega^2 = \omega_0^2 + 2\alpha\theta = 0 + 2 \times 12.5 \times 10 = 250(\text{rad s}^{-1})^2$

Therefore KE gained = $\frac{1}{2} I\omega^2 = \frac{1}{2} \times 0.4 \times 250 = 50$J.

(d) The answers are same, i.e. kinetic energy gained by the wheel = work done by the force. There is no loss of energy due to friction.

44. Prove that the rate of change of total angular momentum of a system of particles about a reference point is equal to the total torque acting on the system.

Sol. Consider a system of n particles, Let $\vec{L}_1, \vec{L}_2, \vec{L}_3$, …. be the angular moments of the various particles about the origin O of a reference frame. The total angular momentum of the system about the point O is given by the vector sum of angular moments of all the individual particles. Thus

$\vec{L} = \vec{L}_1 + \vec{L}_2 + \ldots\vec{L}_n$

$= \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 + \ldots + \vec{r}_n \times \vec{p}_n$

Or $\vec{L} = \sum_{i=1}^{n} \vec{L}_1 = \sum_{i=1}^{n} \vec{r}_1 \times \vec{p}_1$

Similarly the total torque acting on the system is equal to the vector sum of the torques of all the particles about the origin O. Thus

$\vec{\tau}^{total} = \vec{\tau}_1 + \vec{\tau}_2 + \cdots + \vec{\tau}_n = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \ldots + \vec{r}_n \times \vec{F}_n$

Or $\vec{\tau}^{total} = \sum_{i=1}^{n} \vec{\tau}_1 = \sum_{i=1}^{n} \vec{r}_n \times \vec{F}_n$

The total torque acting on the system is due to two sources (i) Torques exerted on the particles of the system by mutual internal forces between the particles (ii) Torques exerted on the individual particles of the system of the external forces.

According to Newton's third law, the internal torque on the system due to internal forces is zero because the forces between any two particles are equal and opposite and directed along the line joining the two particles. Hence the total torque is due to external forces only. So we have $\vec{\tau}^{tot} = \vec{\tau}^{ext} = \sum_{i=1}^{n} \vec{\tau}^{ext}$

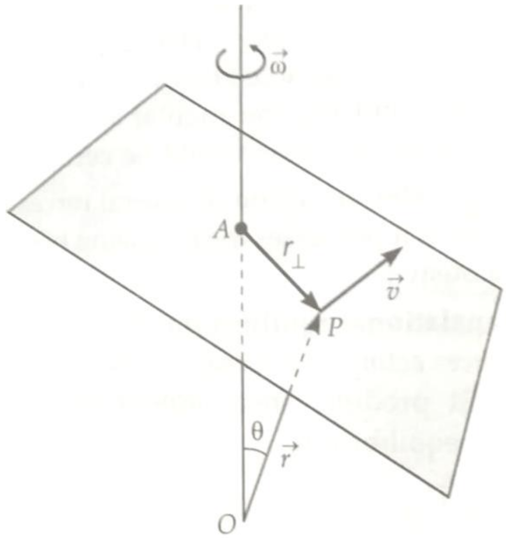
This is accordance with the common experience – bodies do not start spinning on their own without external forces acting on them. Hence if the angular momentum of a system changes with time, this change can be due to the torques produced by external forces only. So we can write $\vec{\tau}^{ext} = \frac{d\vec{L}}{dt}$

Thus the rate of change of total angular momentum of a system of particles about a fixed point is equal to the total external torque acting on the system about that point.

45. Derive an expression for the total work done on a rigid body executing both translational and rotational motions. Hence deduce the condition for the equilibrium of the rigid body.

Sol. Expression for work done in combined rotational and translational motions: A rigid body can have only two kind of motions: translational and rotational. In translational motion, all the particles of the rigid body move with the same velocity v_0, without rotating.

In small time interval, Δt each particle covers a displacement given by $\Delta\vec{s} = \vec{v}_0\,\Delta t$



In rotational motion, every particle of the rigid body rotates about the axis of rotation with the same angular velocity $\vec{\omega}$. As shown in figure we choose the origin O ion the axis of the rotation.

$v = r_\perp \omega = \omega r \sin\theta$

where $r_\perp = r\sin\theta$ is the perpendicular distance of the particle from the axis of rotation and θ is the angle between $\vec{r}$ and the axis of rotation. In vector rotation,

$\vec{v} = \vec{\omega} \times \vec{r}$

The direction of $\vec{v}$ is perependicualr to both $\vec{\omega}$ and $\vec{r}$. the direction of $\vec{\omega}$ is determined buy right hand rule. If we curl the fingers of our right in the direction of rotation of the particle, then the extended thumb gives the direction of the angular velocity vector $\vec{\omega}$.

Displacement covered by the rotating in small time interval Δt

$= \vec{v}\Delta t = \vec{\omega} \times \vec{r}\Delta t = (\vec{\omega}\,\Delta t) \times \vec{r} = \Delta\vec{\phi} \times \vec{r}$

Where $\Delta\vec{\phi} = \vec{\omega}\,\Delta t$ = the small angular displacement0

So, when the rigid body rotates with angular velocity $\vec{\omega}$ and translates with velocity $\vec{v}_0$, the displacement of any point of $\vec{r}$ of the rigid body is given by

$\Delta\vec{r} = \vec{v}_0\Delta t + \vec{\omega}\,\Delta t \times \vec{r} = \Delta\vec{s} + \Delta\vec{\phi} \times \vec{r}$

The work done by an external force $\vec{F}$ acting on point $\vec{r}$ is given by

$\Delta W = \vec{F}\cdot\Delta\vec{r} = \vec{F}.(\Delta\vec{s} + \Delta\vec{\phi} \times \vec{r}) = \vec{F}.\Delta\vec{s} + \vec{F}.(\Delta\vec{\phi} \times \vec{r})$

As the scalar triple product is cyclic, so

$\vec{F}.(\Delta\vec{\phi} \times \vec{r}) = \vec{r}.(\vec{F} \times \Delta\vec{\phi}) = \Delta\vec{\phi}.(\vec{r} \times \vec{F}) = \Delta\vec{\phi}.\vec{\tau} = \vec{\tau}.\Delta\vec{\phi}$

where $\vec{\tau} = \vec{r} \times \vec{F}$ is the torue acting on the particle

So, $\Delta W = \vec{F}.\Delta\vec{s} + \vec{\tau}.\Delta\vec{\phi}$

When a number of force $\vec{F}_i$ act on different points $\vec{r}_1$ of the rigid body, the total work done on the rigid body will be W = $(\Sigma\vec{F}_i) .\Delta \vec{s} + (\Sigma\vec{r}_1).\Delta \vec{\phi}$

For the rigid body to be in equilibrium the work done in the displacement plus rotation should be zero for all choices of $\Delta\vec{s}$ and $\Delta\vec{\phi}$. We therefore obtain the condition

$\Sigma \vec{F}_1 = 0$ and $\Sigma \vec{\tau}_1 = 0$

Hence for a rigid body in equilibrium the sum of the forces acting on it must be zero and the sum of torques acting on it must be zero.

46. Two discs of moments of inertia I_1 and I_2 about their respective axes (normal to the disc and passing through the centre), and rotating with angular speed ω_1 and ω_2 are brought into contact face to face with their axes of rotation coincident, (a) What is the angular speed of the two-disc system? (b) Show that the kinetic energy of the combined system is less than the sum of the initial kinetic energies of the two discs. How do you account for this loss in energy? Take ω_1 not equal to ω_2.

Sol. (a) Let ω be the angular speed of the two disc system. Then by conservation of angular momentum, $(I_1 + I_2)\omega = I_1\omega_1 + I_2\omega_2$

Or $\omega = \frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}$

(b) Initial KE of the two discs, $K_1 = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2$

Final KE of the two disc system, $K_2 = \frac{1}{2}(I_1 + I_2)\omega^2$

$= \frac{1}{2}(I_1 + I_2)\left(\frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}\right)^2$

Loss in KE = $K_1 - K_2 = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2) - \frac{1}{2(I_1+I_2)}(I_1\omega_1 + I_2\omega_2)^2$

$= \frac{1}{2(I_1+I_2)}[I_1^2\omega_1^2 + I_1I_2\omega_2^2 + I_1I_2\omega_1^2 + I_2^2\omega_2^2 - (I_1^2\omega_1^2 + I_2^2\omega_2^2 + 2I_1I_2\omega_1\omega_2)]$

$= \frac{1}{2(I_1+I_2)}[I_1I_2\omega_2^2 + I_1I_2\omega_1^2 - 2I_1I_2\omega_1\omega_2]$

$= \frac{I_1I_2}{2(I_1+I_2)}(\omega_1^2 + \omega_2^2 - 2\omega_1\omega_2)$

$= \frac{I_1I_2}{2(I_1+I_2)}(\omega_1 - \omega_2)^2$ = a positive quantity [Since $\omega_1 \neq \omega_2$]

Hence there is a loss of rotational KE which appears as heat. When the two discs are brought together, work is done against friction between the two discs.

E. ROLLING MOTION

(1 Mark Questions)

1. Work done by friction during pure rolling motion of a ball on Rough surface is zero. (True or False).

Ans. False. The work done depend upon frame of reference. From the frame where lowest point is moving then the work done is zero.

2. Friction is required but necessary for rolling. (True or False).

Sol. True. As Friction provides the necessary tangential force and torque.

(2 Marks Questions)

3. Determine the total kinetic energy of a rolling object.

Sol. For a rolling object, the total kinetic energy is the sum of its rotational kinetic energy and translational kinetic energy. Translational Kinetic Energy: The kinetic energy of an object due to its linear motion is known as translational kinetic energy.

(3 Marks Questions)

4. A cylinder of mass 10kg and radius 15cm is rolling perfectly on a plane of inclination 30°. The coefficient of friction μ = 0.25. (a) How much is the force of friction on acting on the cylinder? (b) What is the work done against friction rolling down? Take $g=10ms^{-2}$.

Sol. Moment of inertia of the cylinder about its geometric axis=1/2 mr^2

Acceleration of the cylinder is given as $a = \frac{mgsin\theta}{m+I/r^2} = \frac{2}{3}gsin30° = 3.27m/s^2$

(a) Using Newton's Second Law, we can write, $f_{net}=ma$

$\Longrightarrow mgsin30° - f = ma$

$\Longrightarrow f = 49N - 32.7N = 16.3N$

(b) During rolling, the instantaneous point of contact with the plane comes to rest. Hence, the work done against frictional force is zero.

5. A solid sphere rolls down two different inclined planes of the same heights but different angles of inclination, (a) Will it reach the bottom with the same speed in each case? (b) Will it take longer to roll down one plane than the other? (c) If so, which one and why?

Sol. Acceleration of the rolling sphere, $a = \frac{g \sin \theta}{(1+\frac{k^2}{R^2})}$

Velocity of the sphere at the bottom of the inclined plane, $v = \sqrt{\frac{2gh}{(1+\frac{k^2}{R^2})}}$

(a) Yes, the sphere will reach the bottom with the same speed v because h is same in both cases.

(b) Yes, the sphere will take longer time to roll down one plane than the other.

(c) The sphere will take larger time in case of plane with smaller inclination because the acceleration $a \propto \sin \theta$.

6. A hoop of radius 2 m weighs 100 kg. It rolls along a horizontal floor so that its centre of mass has a speed of 20 cm/s. How much work has to be done to stop it?

Sol. Here R = 2m, M = 100kg, $v_{cm} = 20\ cms^{-1} = 0.20\ ms^{-1}$

Work required to stop the hoop = Total KE of the hoop

= Rotational KE + translational KE

$= ½ I\omega^2 + ½ mv^2_{cm}$

$= \frac{1}{2} \times MR^2 \times \left(\frac{v_{cm}}{R}\right)^2 + \frac{1}{2} M(v_{cm})^2$

$= Mv^2_{cm} = 100 \times (0.20)^2 = 4J.$

7. A solid cylinder rolls up an inclined plane of angle of inclination 30°. At the bottom of the inclined plane the centre of mass of the cylinder has a speed of 5 m/s.
(a) How far will the cylinder go up the plane?
(b) How long will it take to return to the bottom?

Sol. (a) Total initial kinetic energy of the cylinder, $K_i = \frac{1}{2} Mv^2_{CM} + \frac{1}{2} I_{CM} \omega^2$

$= \frac{1}{2} Mv^2_{CM} + \frac{1}{2} \times \frac{1}{2} MR^2 \times \frac{v^2_{CM}}{R^2}$

$= \frac{1}{2} Mv^2_{CM} + \frac{1}{4} Mv^2_{CM} = \frac{3}{4} Mv^2_{CM}$

Initial potential energy, $U_i = 0$, Final kinetic energy, $K_f = 0$

Final potential energy, $U_f = Mgh = mgs \sin 30° = \frac{1}{2} Mgs$

Where s is the distane travelled up the incline and h is the vertical height covered above the bottom

Gain in PE = Loss in KE

½ <gs = $\frac{3}{4} Mv^2_{CM} = \frac{3v^2_{CM}}{2g} = \frac{3\times(5)^2}{2\times 9.8} = 3.8m$

(b) Using equation of motion for the motion up the m\incline, we get

$0 = v_{CM} + at$ or $a = -\frac{v_{CM}}{t}$

Also, $0^2 - v^2_{CM} = 2as$

Or $a = -\frac{v^2_{CM}}{2s}$

$\therefore \frac{v_{CM}}{t} = \frac{v^2_{CM}}{2s}$

Or $t = \frac{2s}{v_{CM}} = \frac{2\times 3.8}{5} = 1.5s$

Total time taken in returning to the bottom = 2 × 1.5 = 3.0s.

8. Explain why friction is necessary to make the disc roll (refer to Q. 5, Section F) in the direction indicated.
(a) Give the direction of frictional force at B, and the sense of frictional torque, before perfect rolling begins.
(b) What is the force of friction after perfect rolling begins?

Sol. To roll a disc, a torque is required which in turn requires a tangential force to act on it. As the force acting on the disc, so it is necessarily required for the rolling of a disc.

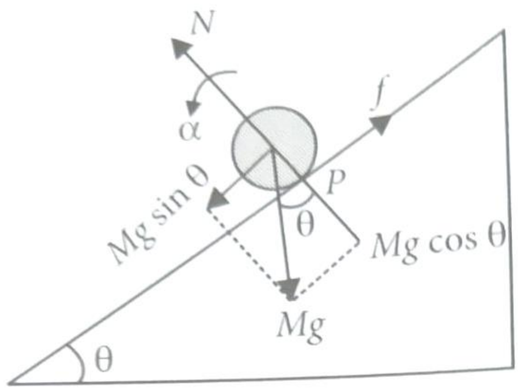
(a) Frictional force at B opposes the velocity of B. Therefore, frictional force is in the same direction as the arrow. The sense of frictional torque is such as to oppose the angular motion. By right hand rule, both $\vec{\omega}_0$ and $\vec{\tau}$ act normal to the plane of paper, $\vec{\omega}_0$ into the plane of paper and $\vec{\tau}$ out of the paper.

(b) Frictional force decreases the velocity of the point of contact B. Perfect rolling begins when this velocity is zero. Once this is, the force of friction is zero.

(5 Marks Questions)

9. Obtain the expression for the linear acceleration of a solid cylinder of radius 'R' rolling down an inclined plane. Also find the frictional force acting between the solid cylinder and the plane.

Sol.



The external forces acting on the cylinder are (i) The weight Mg of the cylinder acting vertically downwards through the centre of mass of the cylinder. (ii) The normal reaction N of the inclined plane acting perpendicular to the plane at P. (iii) The frictional force f acting upwards and parallel to the inclined plane. The weight Mg can be resolved into two rectangular components. (a) Mg cos θ perpendicular to the inclined plane (b) Mg sin θ acting down the inclined plane. As ther is mo m,otion in direction normal to the incline plane, so

$N = Mg \cos\theta$

Applying Newton's second law to the linear motion of the centre of mass, the net force on the cylinder rolling down the inclined plane is

$F = Ma = Mg \sin\theta - f$ …(i)

It is only the force of friction f which exerts torque τ on the cylinder and makes it rotate with angular acceleration α. It acts tangentially at the point of contact P and has lever arm equal to R.

Therefore, τ = Force × Force arm = f.R

Also, τ = MI × angular acceleration = $I\alpha$

So, $fR = I\alpha$ or $f = \frac{I\alpha}{R} = \frac{Ia}{R^2}$ $[\because\ \alpha = \frac{a}{R}]$

From putting the value of f in equation (i) we get

$Ma = Mg \sin\theta - \frac{Ia}{R^2} = \Rightarrow a = g\sin\theta = \frac{Ia}{MR^2}$

Or $a + \frac{Ia}{MR^2} - g\sin\theta$ or $a\left[1 + \frac{1}{MR^2}\right] = g\sin\theta$

$a = \frac{g\sin\theta}{1+\frac{1}{MR^2}}$

Moment of inertia of the solid cylinder about its axis = ½ MR^2

So, $a = \frac{g\sin\theta}{1+\frac{\frac{1}{2}MR^2}{MR^2}}$ or a = 2/3 g sin θ

Clearly the linear acceleration of a solid cylinder rolling down an inclined plane is less that the acceleration due to gravity g(a < g). The linear acceleration of the cylinder is constant for a given inclined plane (or given θ) and is independent of the mass M and R.

The value of force of friction is $F = Mg \sin\theta - Ma$
$= Mg \sin \theta - M.\frac{2}{3} g \sin \theta = \frac{1}{3}Mg \sin \theta$.

10. Read each statement below carefully, and state, with reasons, if it is true or false:
(a) During rolling, the force of friction acts in the same direction as the direction of motion of the CM of the body.
(b) The instantaneous speed of the point of contact during rolling is zero.
(c) The instantaneous acceleration of the point of contact during rolling is zero.
(d) For perfect rolling motion, work done against friction is zero.
(e) A wheel moving down a perfectly frictionless inclined plane will undergo slipping (not rolling) motion.

Sol. (a) True. When a body rolls, the force of friction acts in the same direction as the direction of motion of the CM of the body.

(b) True. A rolling body can be imagined t the rotating about an axis passing through the point of contact of the body and the ground. Hence the instantaneous speed of the point of contact is zero.

(c) False. As the body is rotating, its instantaneous acceleration cannot be zero.

(d) True. Perfect rolling begins when force of friction is zero. So work done agains friction is zero.

(e) True. On a perfectly frictionless inclined plane, there is no tangential force of friction. So the wheel cannot roll. It will simply slip under the effect of its own weight.

F. Case Study

1. Quantization: In Physics some measurable quantities are quantized. A physical quantity is said to be quantized if it can have only discrete (not continuous) values. Some examples of quantities which are quantized are mass, charge and energy. For sub atomic particles (called fundamental particle) the angular momentum is quantized, Fundamental particles such as electrons and protons have a certain intrinsic angular momentum of their own. The angular momentum is called spin angular momentum. The spin angular momentum of a fundamental particle is quantized and its value is given by $S = n\frac{h}{2\pi}$ where h is the Planck's constant = 6.63×10^{-34} HS and n is a number called the spin quantum number. The value of n for electrons, protons, positrons and antiprotons can be + ½ and – ½. Pions have n = 0.

(i) Which of the following is not quantized?
(a) Mass (b) Energy (c) Linear momentum (d) charge

Sol. The correct choice is c.

(ii) Choose the correct statements from the following:
(a) Electric charge on a charged body can only be an integral multiple of the smallest possible charge
(b) Energy can have only discrete values

(c) The spin angular momentum of an electron can be + h/4π or – h/4π.

(d) The spin angular momentum of a pion is + h/4π or – h/4π.

Sol. The correct choices are a, b and c. The spin angular momentum of a pion is zero.

G. ASSERTION REASON TYPE QUESTIONS

(a) If both assertion and reason are true and reason is the correct explanation of assertion.
(b) If both assertion and reason are true but reason is not the correct explanation of assertion.
(c) If assertion is true but reason is false **(d) If both assertion and reason are false**
(e) If assertion is false but reason is true

1. Assertion: When a body is dropped from a height explodes in mid air, but its centre of mass keeps moving in vertically downward direction.

   Reason: Explosion occur under internal forces only. External force is zero.

Ans. (a) Both assertion and reason are true and reason is the correct explanation of assertion. Explosion is due to internal forces. As no external force is involved, the vertical downward motion of centre of mass is not affected.

2. Assertion: Power associated with torque is product of torque and angular speed of the body about the axis of rotation.

   Reason: Torque is equal to rate of change of angular momentum.

Ans. (b) Both assertion and reason are true but reason is not the correct explanation of assertion.

Power = Work done/time. In rotatory motion torque is analogous to force.

$\therefore P = \frac{\tau\theta}{t} = \tau \times \omega$. In rotatory motion.

P = $\frac{W}{t} = F \times v$. In translator motion.

3. Assertion: A helicopter must necessarily have two propellers.

   Reason: Two propellers are provided in helicopter in order to conserve linear momentum.

Ans. (c) Assertion is true but reason is false.

If there are only one propeller in the helicopter, the helicopter itself, would have turned in opposite direction of the direction of propeller due to conservation of angular momentum. Thus two propellers provide helicopter a steady movement.

4. Assertion: A person standing on a rotating platform suddenly stretched his arms, the platforms slows down.

   Reason: A person by stretching his arms increases the moment of inertia and decreases angular velocity.

Ans. (a) Both assertion and reason are true and reason is the correct explanation of assertion.

On stretching arms, his moment of inertia (- MR^2) increases. As no torque has been applied, therefore, from conservation of angular momentum, Iω = constant. As I increases, ω decreases, i.e., the platform slows down.

5. Assertion: The centre of mass of uniform triangular lamina is centroid.

Reason: Centroid is centre of symmetry of mass of the triangular lamina.

Ans. (a) Both assertion and reason are true and reason is the correct explanation of assertion.

H. CHALLENGING PROBLEMS

1. (a) A child stands at the centre of a turntable with his arms outstretched. The turntable is set rotating with an angular speed of 40 rev/min. How much is the angular speed of the child if he folds his hands back and thereby reduces his moment of inertia to 2/5 times the initial value? Assume that the turntable rotates without friction, (b) Show that the child's new kinetic energy of rotation is more than the initial kinetic energy of rotation. How do you account of this increase in kinetic energy?

Sol. (a) Here ω_1 = 40rpm, $l_2 = 2/5\ l_1$

By the principal of conservation of angular momentum,

$l_1\omega_1 = l_2\omega_2$ or $l_1 \times 10 = 2/5\ l_1\omega_2$ or ω_2 = 100 rpm.

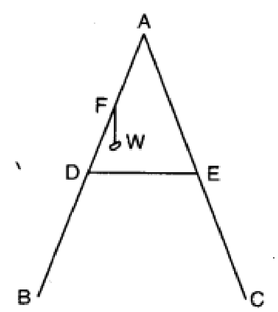
(b) Initial kinetic energy of rotation, $½\ l_1\omega_1^2 = ½\ l_1(40)^2 = 800l_1$

Now kinetic energy of rotation= $½\ l_2\omega_2^2 = ½ \times 2/3\ l_1 = (100)^2 = 2000l_1$

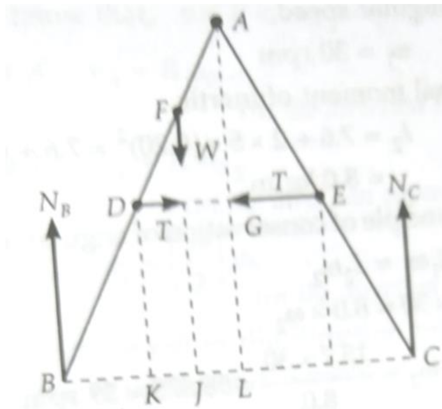
$\therefore \frac{\text{New KE}}{\text{Inital KE}} = \frac{2000\ l_1}{800\ l_1}$ = 2.5.

Thus the child's new kinetic energy of rotation is 2.5 times its initial kinetic energy of rotation. This increase in kinetic energy is due to the internal energy of the child which he used in folding his hands back from the out stretched position.

2. As shown in Fig. the two sides of a step ladder BA and CA are 1.6 m long and hinged at A. A rope DE, 0.5 m is tied halfway up. A weight 40 kg is suspended from a point F, 1.2 m from B along the ladder BA. Assuming the floor to be friction less and neglecting the weight of the ladder, find the tension in the rope and forces exerted by the floor on the ladder. (Take g = 9.8 m^2)(Hint: Consider the equilibrium of each side of the ladder separately.)



Sol. A shown in figure below, N_B and N_C be the normal reactions of the floor at B and C respectively and R be the tension in the rope DE. Then



$N_B + N_C = W = 40 \times 9.8N$
Or N_B + N=C= = 392N …(1)
Consider the portion AC of the ladder. Balancing torques about A, we get
$N_C \times LC = T \times AG$
From the geometry of the figure we get LC = 2GE = DE = 0.5m or GE = 0.25m
$\therefore AG = \sqrt{AE^2 - GE^2} = \sqrt{(0.8)^2 - (0.25)^2} = \sqrt{0.5775} = 0.76m$
Hence $N_C \times 0.5 \times T \times 0.76$ or $T = 0.66\ N_C$ …(2)
Now consider the portion AB of the ladder. Balancing the torques about A, we get
$N_B \times BL - W \times JL = T \times AG$
But JL = ¼ DE = ¼ × 0.5 = 0.125m
BL = DE = 0.5m, W = 392N
$\therefore N_B \times 0.5 = 392 \times 0.125 = T \times 0.76$
Or $N_B \times 0.5 - 392 \times 0.125 = 0.66\ N_C \times 0.76$ [Using (2)]
Or $N_B - 98 = N_C$
Or $N_B - N_C = 98$ …(3)
On solving (1) and (3) we get
$N_B = 245N$, $N_C = 245 - 98 = 147N$ and $T = 0.66\ N_C = 0.66 \times 147 = 97N$.

3. A man stands on a rotating platform, with his arms stretched horizontally holding a 5 kg weight in each hand. The angular speed of the platform is 30 revolutions per minutes. The man then brings his arms close to his body with the distance of each weight from the axis changing from 90 cm to 20 cm. The moment of inertia of the man together with the platform may be taken to be constant and equal to 7.6 kg m^2.(a) What is his new angular speed? (Neglect friction)(b) Is kinetic energy conserved in the process? If not, from where does the change come about?

Sol. (a) Total initial moment of inertia,
I_1 = MI of man and platform + MI of two 5kg weights
$= 7.6 + 2 \times 5 \times (0.90)^2 = 7.6 + 8.1 = 15.7$ kg m^2
Initial angular speed, $\omega_1 = 30$ rpm
Total final moment of inertia, $I_2 = 7.6 + 2 \times 5 \times (0.20)^2 = 7.6 + 0.4 = 8.0$ kg m^2
By the principle of conservation of angular momentum,
$I_1\omega_1 = I_2\omega_2$
Or $15.7 \times 30 = 8.0 \times \omega_2$
Or $\omega_2 = \frac{15.7Ø30}{8.0} = 58.873 = 59$ rpm.

(b) $\frac{\text{Final KE}}{\text{Initial KE}} = \frac{\frac{1}{2}I_1\omega_1^2}{\frac{1}{2}I_2\omega_2^2} = \frac{8.0\times(59)^2}{15.7\times(30)^2} = 1.97$

Thus the final K.E. is about twice the initial K.E. i.e, KE is not conserved in the process. The increase in KE is due to the internal energy the man uses in bringing his arms closer to his body.

4. A bullet of mass 10 g and speed 500 m/s is fired into a door and gets embedded exactly at the centre of the door. The door is 1.0 m wide and weighs 12 kg. It is hinged at one end and rotates about a vertical axis practically without friction. Find the angular speed of the door just after the bullet embeds into it.(Hint: The moment of inertia of the door about the vertical axis at one end is $ML^2/3$.)

Sol. By the principal of conservation o angular momentum,

Initial angular momentum of the bullet = Final angular momentum of the door

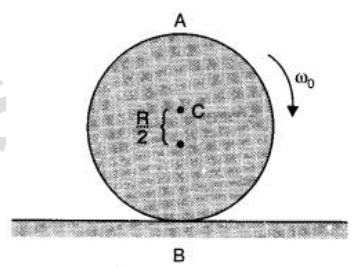
Or $\mu r = I\omega$

Or $mvr = \frac{ML^2}{3} \times \boldsymbol{\omega}$ or $\omega = \frac{3mvr}{ML^2}$

Here $m = 10g = 10^{-2}kg$, $L = 1.0m$, $M = 12kg$

$\therefore \omega = \frac{3\times10^{-2}\times500\times0.5}{12\times(1.0)^2} = 0.625$ rad s^{-1}.

5. A disc rotating about its axis with angular speed w_o is placed lightly (without any translational push) on a perfectly friction less table. The radius of the disc is R. What are the linear velocities of the points A, B and C on the disc shown in Fig.? Will the disc roll in the direction indicated?



Sol. We know that $v = R\omega$

For point A : $v_A = R\omega_0$ (in the direction of the arrow)

For point B: $v_B = R\omega_0$ (in the direction opposite to arrow)

For point C: $v_C = (R/2)\omega_0$ (in the direction of the arrow)

The disc will roll in the direction indicated. It is because the disc is placed on a perfectly frictionless table and without friction, a body cannot roll.

6. A solid disc and a ring, both of radius 10 cm are placed on a horizontal table simultaneously, with initial angular speed equal to 10π rad/s. Which of two will start to roll earlier? The coefficient of kinetic friction is $u_k = 0.2$.

Sol. In case of pure rotation without translation, the velocity of centre of mass is zero. The friction reduces the speed at the point of contact and as such accelerates the centre of mass till the velocity of centre of mass becomes equal to v = Rω and the instantaneous velocity at the contact point becomes zero. Thus the force of friction $\mu_k mg$ produces an acceleration a in the centre of mass. So the equation f motion for CM is

$\mu_k mg = ma$ …(1)

The torque due to force of friction is $\mu_k mg \times R$. It produces angular retardation given by

$\mu_k mg = - I \alpha$ …(2)

Rolling begins when v = Rω

But $v = 0 + at = \mu_k gt$ …(3) [From (1), $a = \mu_k gt$]

And $\omega = \omega_0 + \alpha t = \omega_0 - \frac{\mu_k mgR}{I}t$ [using (2)

Or $\frac{v}{R} = \omega_0 - \frac{\mu_k mgR}{I}t$

Or $\frac{\mu_k gt}{R} - \omega_0 - \frac{\mu_k mgR}{I}t$

Or $\frac{\mu_k gt}{R}\left[1 + \frac{mR^2}{I}\right] = \omega_0$

Or $t = \frac{R\omega_0}{\mu_k g\left(1+\frac{mR^2}{I}\right)}$

For a disc: $I = mR^2/2$

$\therefore I = \frac{R\omega_0}{3\mu_k g} = \frac{0.10\times10\pi}{3\times0.2\times9.8} = 0.53s$

For a ring: $I = mR^2$

$\therefore I = \frac{R\omega_0}{2\mu_k g} = \frac{0.10\times10\pi}{2\times0.2\times9.8} = 0.80s$

Thus the disc begins to roll earlier than the ring.

7. A cylinder of mass 10 kg and radius 15 cm is rolling perfectly on a plane of inclination 30°. The coefficient of static friction us = 0.25.
(a) How much is the force of friction acting on the cylinder?
(b) What is the work done against friction during rolling?

Sol. Mass of cylinder, M = 10kg, Radius of cylinder, R = 15cm = 0.15m, Angle of inclination, θ = 30°, Coefficient of friction, μ = 0.25

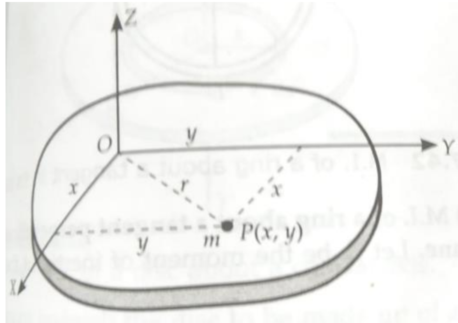
(a) Force of friction acting on the cylinder, $f = \frac{1}{3}Mg\sin\theta = \frac{1}{3} \times 10 \times 10 \times \sin 30° = 16.66N$

(b) The work done against friction during rolling is zero because the condition of rolling without slipping is that each instant the line of contact of the cylinder with the surface ios momentarily at rest and the cylinder rotates about this line as axis. The centre of mass of the cylinder moves in a straight line parallel to the inclined plane. So it is friction which prevents sliping.

8. (a) Prove the theorem of perpendicular axes.
Hint: Square of the distance of a point (x, y) in the x-y plane from an axis through the origin perpendicular to the plane is $x^2 + y^2$]
(b) Prove the theorem of parallel axes.
Hint: If the centre of mass of chosen the origin [$\Sigma m_i r_i = 0$]

Sol. (a) Theorem of perpendicular axes:
Proof: Consider a plane lamina lying in the XOY plane, it can be assumed to be made up of large number of particles. Consider one such particle of mass m situated at point P(x,

y). Clearly, the distance of the particle from X, Y and Z axes are y, x and rn respectively such that $r^2 = y^2 + x^2$



Moment of inertia f the particle about X axis = my^2

$\therefore$ Moment of inertia of whole lamina about X axis is $I_x = \Sigma\, my^2$

Moment of inertia of whole lamina about Y axis is $I_y = \Sigma\, mx^2$

Moment of inertia of whole lamina about Z axis is $I_x = \Sigma\, mr^2 = \Sigma\, m(y^2 + x^2)$

$= \Sigma\, my^2 + \Sigma\, mx^2$

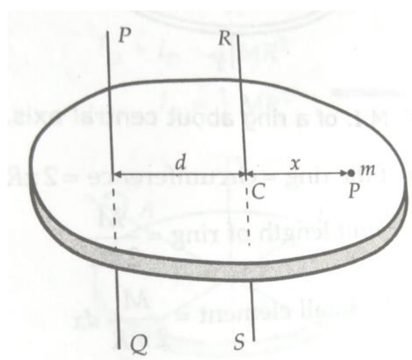
Or $I_z = I_x + I_y$

This proves the theorem of perpendicular axis

(b) Theorem of parallel axes:

Proof: Let I be the moment of inertia of a body of mass M about an axis PQ. Let RS be a parallel axis passing through the centre of mass C of the body and at distance d from PQ. Let I_{CM} be the moment of inertia of the body about the axis RS.

Consider a particle P of mass m at a distance x from RS and so at distance (x +d) from PQ.



Moment of inertia of the particle about axis PQ = $m(x + d)^2$

$\therefore$ Moment of inertia of the whole body about the axis PQ is

$I = \Sigma\, m(x + d)^2 = \Sigma\, m(x^2 + d^2 + 2xd)$

Now $\Sigma\, mx^2 = I_{CM}$

$\Sigma\, md^2 = (\Sigma\, m)d^2 – Md^2$

$\Sigma\, 2\, mxd = 2d\, \Sigma\, mx = 2d \times 0 = 0$

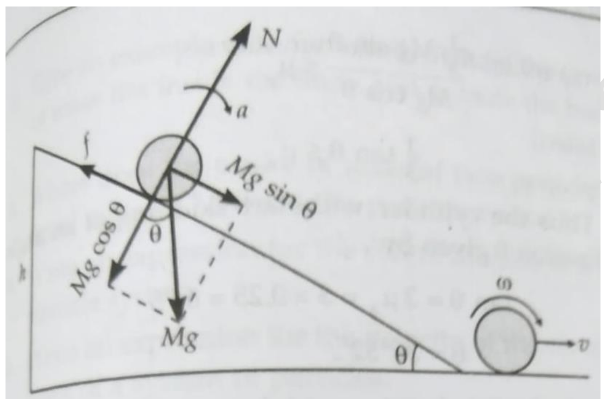
This is because a body can balance itself about its centre of mass so the algebraic sum of moments (Σ mx) of masses of all the particles about the axis RS is zero.

Hence $I = I_{CM} + Md^2$

This proves the theorem of parallel axis.

9. Prove the result that the velocity v of translation of a rolling body (like a ring, disc, cylinder or sphere) at the bottom of an inclined plane of a height h is given by,$v^2=2gh/(1+k^2/R^2)$ using dynamical consideration (i.e., by consideration of forces and torques). Note k is the radius of gyration of the body about its symmetry axis, and R is the radius of the body. The body starts from rest at the top of the plane.

Sol. Velocity attained by a body rolling down on inclined plane" Consider a body of mass M and radius R rolling down a plane inclined at an angle θ with the horizontal as shown in figure. It is only due to friction at the line of contact that body can roll without slipping. The centre of mass of the body moves in a straight line parallel to the inclined plane.



The external forces on the body are
(i) The weight Mg acting vertically downwards.
(ii) The normal reaction N of the inclined plane.
(iii) The force of friction acting up the inclined plane.

Let a be the downward acceleration of the body, The equations of motion for the body can be written as N – Mg cos θ = 0

F = Ma = Mg sin q - f

As the force of friction provides the necessary torque for rolling so

$\tau = f \times R = I\alpha = MR^2 \left(\frac{a}{R}\right)$

Or $f = M\frac{k^2}{R^2}.a$

Where k is the radius of gyration of the body about its axis of rotation. Clearly

$Ma = Mg \sin\theta = M\frac{k^2}{R^2}.a$

Or $a = \frac{g\sin\theta}{(1+\frac{k^2}{R^2})}$

Let h be the height of the inclined plane and s the distance travelled by the body down the plane. The velocity v attained by the body at the bottom of the inclined plane can be obtained as follows:

$v^2 - u^2 = 2as$

or $v^2 - 0^2 = 2.\frac{g\sin\theta}{(1+\frac{k^2}{R^2})}.s$

or $v^2 = \frac{2gh}{1+k2/R2}$ [since h/s – sinθ]

or $v = \sqrt{\frac{2gh}{(1+\frac{k^2}{R^2})}}$

SPACE FOR ROUGH WORK

SPACE FOR NOTES